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Then,

$$k = \frac{xy}{2}. \quad (1)$$

Now, $P = AC + CB + AB$, and for a minimum $dP/dy = 0$ and $\delta P/\delta \alpha = 0$. But, $AC = y/\sin \alpha = y \csc \alpha$; and since $\overline{CB}^2 = \overline{CD}^2 + \overline{DB}^2$ and $DB = AB - AD = x - y \cot \alpha$,

$$CB = [y^2 + (x - y \cot \alpha)^2]^{\frac{1}{2}} = \left(y^2 \csc^2 \alpha + \frac{4k^2}{y^2} - 4k \cot \alpha \right)^{\frac{1}{2}}.$$

Then

$$P = y \csc \alpha + \left(y^2 \csc^2 \alpha + \frac{4k^2}{y^2} - 4k \cot \alpha \right)^{\frac{1}{2}} + \frac{2k}{y}.$$

$$\therefore \frac{dP}{dy} = \csc \alpha + \frac{1}{2} \left(y^2 \csc^2 \alpha + \frac{4k^2}{y^2} - 4k \cot \alpha \right)^{-\frac{1}{2}} \left(2y \csc^2 \alpha - \frac{8k^2}{y^3} \right) - \frac{2k}{y^2} = 0,$$

which can be written

$$(y^2 \csc \alpha - 2k)[(y^4 \csc^2 \alpha + 4k^2 - 4ky^2 \cot \alpha)^{-\frac{1}{2}} - (2k + y^2 \csc \alpha)] = 0.$$

Hence,

$$y^2 \csc \alpha - 2k = 0 \quad (2)$$

and

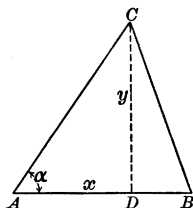
$$y^4 \csc^2 \alpha + 4k^2 - 4ky^2 \cot \alpha = (2k + y^2 \csc \alpha)^2,$$

which reduces to

$$-4ky^2 \cot \alpha = 4ky^2 \csc \alpha.$$

Hence,

$$-\cot \alpha = \csc \alpha \quad \text{or} \quad \alpha = 180^\circ.$$



Substituting (1) in (2) we have $y^2 \csc \alpha = xy$ or $\sin \alpha = y/x$; and since $\sin \alpha = y/AC$, $AC = x$, that is, $AC = AB$. Also

$$dP/d\alpha = -y \csc \alpha \cot \alpha + \frac{1}{2} (y^2 \csc^2 \alpha + 4k^2/y^2 - 4k \cot \alpha)^{-\frac{1}{2}} (-2y^2 \csc^2 \alpha \cot \alpha + 4k \csc^2 \alpha) = 0;$$

or reducing,

$$\cot \alpha \cdot (y^4 \csc^2 \alpha + 4k^2 - 4ky^2 \cot \alpha)^{\frac{1}{2}} = 2k - y^2 \cot \alpha.$$

Squaring both sides and reducing, we get

$$(y^2 \cot \alpha - k)(1 - \cos^2 \alpha) = 0.$$

Hence,

$$1 - \cos^2 \alpha = 0 \quad \text{and} \quad y^2 \cot \alpha - k = 0. \quad (3)$$

Solving (2) and (3) for α , we get $\tan \alpha = 2 \sin \alpha$. Hence $\alpha = 60^\circ$; and since we have shown above that $AB = AC$ and $\alpha = 60^\circ$ is the included angle between these two lines, it follows that $CB = AB = AC$.

Also solved by FRANK IRWIN.

391. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $0 < \lambda < 1$ and $0 < x < \pi$ show that the function $(\sin \lambda x)/(\sin x)$ increases as x increases.

I. SOLUTION BY H. S. UHLER, Yale University.

Applying the well-known formula

$$\sin \theta = \theta \left[1 - \left(\frac{\theta}{\pi} \right)^2 \right] \left[1 - \left(\frac{\theta}{2\pi} \right)^2 \right] \left[1 - \left(\frac{\theta}{3\pi} \right)^2 \right] \cdots, \quad [\theta^2 < \infty],$$

to the problem under consideration we get

$$\frac{\sin \lambda x}{\sin x} = \frac{\lambda \left[1 - \left(\frac{\lambda x}{\pi} \right)^2 \right] \left[1 - \left(\frac{\lambda x}{2\pi} \right)^2 \right] \left[1 - \left(\frac{\lambda x}{3\pi} \right)^2 \right] \cdots}{\left[1 - \left(\frac{x}{\pi} \right)^2 \right] \left[1 - \left(\frac{x}{2\pi} \right)^2 \right] \left[1 - \left(\frac{x}{3\pi} \right)^2 \right] \cdots}.$$

Consider now the ratio of the n th binomial factor of the numerator to the n th or corresponding factor of the denominator:

$$\frac{1 - \left(\frac{\lambda x}{n\pi} \right)^2}{1 - \left(\frac{x}{n\pi} \right)^2} = \frac{n^2\pi^2 - \lambda^2 x^2}{n^2\pi^2 - x^2} = 1 + \frac{(1 - \lambda^2)x^2}{n^2\pi^2 - x^2}.$$

For any chosen value of n (1, 2, 3, ...) the denominator of the last fraction decreases while the numerator increases as x grows larger. The fraction is always finite and positive because of the hypotheses $0 < \lambda < 1$ and $0 < x < \pi$. Consequently, the ratio of any binomial in the numerator of the expression for $(\sin \lambda x)/(\sin x)$ to the corresponding binomial in the denominator increases with x . It is accordingly manifest that the product of an infinite number of such converging ratios increases as x increases, and so the problem is solved.

II. SOLUTION BY FRANK IRWIN, University of California.

We shall show that, for the values of x in question, the derivative of the function is positive. This derivative is

$$\frac{\lambda \cdot \cos \lambda x \cdot \sin x - \sin \lambda x \cdot \cos x}{\sin^2 x}$$

or

$$\frac{1}{x \cdot \sin^3 x \cdot \sin \lambda x} (\lambda x \cdot \cot \lambda x - x \cdot \cot x).$$

This will be positive if $\lambda x \cdot \cot \lambda x > x \cdot \cot x$, that is, if $y \cdot \cot y$, let us say, continually decreases as y increases from 0 to π . This is so, since its derivative, $\cot y - y \csc^2 y$, or $(\sin 2y - 2y)/2 \sin^2 y$, is always negative.

A solution similar to the second was received from ELIJAH SWIFT.

392. Proposed by HORACE OLSON, Student at The University of Chicago.

Two right cylinders of radii a and b , respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Assume $a > b$. Let the axis of the smaller cylinder be the z -axis, that of the larger, the y -axis. We see that the volume is

$$V = 8 \int_0^b \int_0^{\sqrt{b-x^2}} \sqrt{a^2 - x^2} \cdot dy \cdot dx = 8 \int_0^b \sqrt{(a^2 - x^2)(b^2 - x^2)} \cdot dx.$$

This latter integral is elliptic, and may be expressed in terms of complete elliptic integrals of the first and second kinds. Letting $x = b \operatorname{sn}(y, b/a)$, the integral becomes

$$8ab^2 \int_0^K \left\{ 1 - \frac{a^2 + b^2}{a^2} \operatorname{sn}^2 y + \frac{b^2}{a^2} \operatorname{sn}^4 y \right\} dy.$$

This, in turn, may be integrated by reduction formulas and gives finally for the volume,

$$V = \frac{8}{3} a \left[(b^2 - a^2) K \left(\frac{b}{a} \right) + (a^2 + b^2) E \left(\frac{\pi}{2}, \frac{b}{a} \right) \right],$$

where K denotes the complete elliptic integral of the first kind, E the elliptic integral of the second kind.